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ON STABILITY OF CERTAIN NONSTATIONARY SYSTEMS*

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Asymptotic stability of motion of mechanical objects the parameters of the mathematical models of which undergo sharp changes with respect to time, are investigated. An artificial satellite in an orbit with certain masses attaching themselves to or being detached from it is an example of such an object. This gives rise to the problem of choosing a mode of variation of the mathematical models under which the system would remain asymptotically stable.

Let the mechanical motion of an object be described by the following system of differential equations:

$$dx/dt = A(\beta)x, \quad t \geqslant t_0 \tag{1}$$

Here x is the phase vector of dimension n; β is an unknown vector function the components $\beta_i = \beta_i(t)$ $(i = 1, 2, \ldots, l)$ of which are piecewise constant functions continuous from the right at the points of discontinuity and assuming some values from the closed intervals $[a_i, b_i]$ for $t \geqslant t_0$, where a_i and b_i are known constants; $A(\beta) = \{a_{ij}(\beta)\}_{i,j=1}^n$ is an $n \times n$ matrix with components continuously differentiable in β.

Under the above assumptions the matrix $A(\beta(t))$ is piecewise constant in t for all $t \geqslant t_0$, since there exists a unique continuous solution of the Cauchy problem for the system (1) on any realization of $\beta(t)$.

Let $\Omega(t_0, T)$ be a set of bounded, piecewise constant vector functions

$$\Omega(t_0, T) = \{g(t): g(t) = (g_1(t), \ldots, g_l(t)), -\infty < a_i \leqslant g_i(t) \leqslant b_i < \infty,$$
$$t \in [t_0, T], \; i = 1, 2, \ldots, l\}$$

Theorem. The unperturbed motion $x(t) \equiv 0$ of the system (1) is asymptotically stable, it is necessary and sufficient that an instant of time $T \geqslant t_0$ exists such that the condition

$$\max_{\beta \in \Omega(t_0, T)} \sum_{l, k=1}^{n} X_{lk}^2(\beta, T, t_0) < 1 \tag{2}$$

holds, where X_{lk} are components of the fundamental matrix of solutions $X(\beta, T, t_0)$ of the system (1).

The sufficiency of the conditions of the theorem can be proved in the same manner as the proof of Theorem 4 of /1/.

Necessity. Let the unperturbed motion $x(t) \equiv 0$ of the system (1) be asymptotically stable, without however assuming that a finite $T > t_0$ exists for which condition (2) holds. Then, for every $\tau > t_0$ there exists at least one vector function $\beta(t)$, $t \in [t_0, \tau]$, such that

$$\sum_{i, j=1}^{n} X_{ij}^2(\beta, \tau, t_0) \geqslant 1 \tag{3}$$

If $x(t_0) \in G_\lambda = \{x: x^* x \leqslant \lambda^2\}$ for $t = t_0$, then for $t = \tau$ we have

$$x(\tau) \in Q = \{x: x^* R(\beta, \tau, t_0) x \leqslant \lambda^2\}$$
$$R(\beta, \tau, t_0) = X^{-1*}(\beta, \tau, t_0) X^{-1}(\beta, \tau, t_0)$$

The set Q represents, in the phase space, an ellipsoid for which the following relations hold /2/:

$$\max_{x \in Q} (e_i^* x)^2 = e_i^* R^{-1}(\beta, \tau, t_0) e_i \lambda^2 \quad (i = 1, 2, \ldots, l)$$

where e_i is a unit vector. Obviously,

$$\sum_{i, j=1}^{n} X_{ij}^2(\beta, \tau, t_0) = \sum_{i=1}^{n} R_{ii}^{-1}(\beta, \tau, t_0)$$

Therefore from (3) follows

$$\sum_{i=1}^{n} \max (e_i^* x)^2 = \lambda^2 \sum_{i=1}^{n} R_{ii}^{-1}(\beta, \tau, t_0) \geqslant \lambda^2$$

Thus an index i $(1 \leqslant i \leqslant n)$ exists such that

$$\max_{x \in Q} (e_i x) \geqslant \frac{\lambda}{n}$$

This contradicts, by virtue of the arbitrariness of $\tau > t_0$, the assumption that the solution $x(t) \equiv 0$ of (1) is asymptotically stable.

Let us discuss the feasibility of practical application of the conditions of the theorem. It can be shown that the matrix $R^{-1}(\beta, t, t_0)$ is a solution of the Cauchy problem

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$$\frac{dR^{-1}(\beta, t, t_0)}{dt} = A(\beta) R^{-1}(\beta, t, t_0) + R^{-1}(\beta, t, t_0) A^*(\beta) \qquad (4)$$
$$R^{-1}(\beta, t_0, t_0) = E$$

Therefore we can reduce the problem of verifying the conditions of the theorem to the following problem of optimal control: to find an optimal vector function $\beta(t)$ and a least necessary time $T > t_0$ minimizing the functional

$$J(\beta, T) = 1 \Bigg/ \sum_{i=1}^{n} R_{ii}^{-1}(\beta, T, t_0) \qquad (5)$$

Let us introduce the transformation /3/

$$t = \omega\tau + t_0, \quad 0 \leqslant \tau \leqslant 1$$

and write

$$S^* = (R_{11}, \ldots, R_{n1}, R_{12}, \ldots, R_{n2}, \ldots, R_{nn})$$
$$C^* = (1, 0, 0, \ldots, 0, 0, 1, 0, \ldots, 0, \ldots, 0, 0, \ldots, 1)$$
$$B(\beta) = \{G_{ij}(\beta)\}_{i,j=1}^{n}$$
$$G_{ij}(\beta) = \begin{cases} A(\beta) + a_{ii}(\beta) E, & i = j \\ a_{ij}(\beta) E, & i \neq j \end{cases}$$

This enables us to consider, instead of the problem (4), (5), the problem of optimal control with additional control parameter ω, but with fixed time of control, for the following system of scalar equations

$$\min_{\omega} \min_{\beta \in \Omega(0,1)} [C^* S(1)]^{-1} \qquad (6)$$
$$\frac{dS}{d\tau} = \omega B(\beta) S, \quad S(0) = C$$

We can solve (6) using the following gradient method:

$$\beta^{i+1}(\tau) = \mathrm{Pr}_{\Omega(0,1)} \left(\beta^i(\tau) + \alpha_1 \frac{\partial H}{\partial \beta}\right)$$
$$\omega^{i+1} = \omega^i + \alpha_2 \int_0^1 \frac{\partial H}{\partial \omega} d\tau$$

where α_1 and α_2 are positive scalar constants obtained, just as in the gradient method of quickest descent, by minimizing the function $J(\beta^{i+1}, \omega^{i+1})$; $H(S, \beta, \psi, \omega) = \omega\psi^* BS$ is a Hamiltonian, $\psi(\tau)$ is the vector of the conjugate system

$$d\psi/d\tau = -\partial H/\partial S, \quad \psi(1) = C$$

and $\mathrm{Pr}\,\Omega(0,1)$ is the projection operator on the set $\Omega(0, 1)$.

The convergence of the gradient method depends on the type of the relation $A(\beta(t))$ and was investigated in /3,4/.

If the process of solving the problem formulated above yields $\beta(t)$ and ω which supply the global minimum to the functional (6), then (according to the conditions of the theorem) we can arrive at an unambiguous conclusion concerning the asymptotic stability of the unperturbed motion of the system (4). Unfortunately, all known methods of solving the problems of optimal control of the type (6) clarify only the local behavior of the functionals. Therefore, only the sufficient conditions of the stability can be verified in practice.

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